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On the Connection of Posse's L_1- and Zolotarev's Maximum-Norm Problem

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In this paper it is demonstrated that the solution of Posse's problem, i.e., to describe the numbers $a, b \in \mathbb{R}$, $1 < a < b$, for which the functional $\int_{-1}^{1} |P_n| + (-1)^n \int_a^b P_n$ attains its minimum on the set of polynomials of degree n with leading coefficient one, is implicitly contained in the solution of Zolotarev's problem.

1. INTRODUCTION AND NOTATION

In 1878 Zolotarev solved the following problem, now called the Zolotarev problem (see, e.g., the book of Achieser [1, pp. 303–308] or the expository paper [3] of Carlson and Todd): Let $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. *Among all polynomials of the form* $x^n - n\sigma x^{n-1} + a_2 x^{n-1} + \cdots + a_n$, *where* $\sigma \in \mathbb{R}$ *is given and* $(a_2, ..., a_n) \in \mathbb{R}^{n-1}$, *find the one which deviates least from zero on* $[-1, 1]$ *in the maximum norm.* The minimal polynomial is nowadays called the Zolotarev polynomial. While the Zolotarev polynomial can be determined easily when $0 \leqslant |\sigma| \leqslant \tan^2(\pi/2n)$—it is the transformed Tchebycheff polynomial of the first kind $2^{1-n}(1+\sigma)^n T_n((x-\sigma)/(1+\sigma))$—the explicit description of the Zolotarev polynomial for the remaining case, $|\sigma| > \tan^2(\pi/2n)$, is very complicated. A first investigation, using the Equal Ripple Theorem, gives that each monic polynomial $\hat{Z}_n$ of degree n which is a Zolotarev polynomial for some σ with $|\sigma| > \tan^2(\pi/2n)$ can be described completely in the following way: Put $Z_n(x) := \hat{Z}_n(x)/\max_{x \in [-1,1]} |\hat{Z}_n(x)|$. There exist points $y_2, ..., y_{n-1}, a, b \in \mathbb{R}$ with $-1 =: y_1 < y_2 < \cdots < y_{n-1} < y_n := 1 < a < b$ such that the following properties hold:

(i) $\max_{x \in [-1,1] \cup [a,b]} |Z_n(x)| \leqslant 1$, and

(ii) $Z_n(y_j) = (-1)^{n+1-j}$ for $j = 1, ..., n$, and $Z_n(b) = -Z_n(a) = 1$;

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i.e., Z_n consists on $[-1, 1]$ of $(n-1)$ monotonic arcs varying between 1 and -1, $Z_n(x) < -1$ on $(1, a)$, and on $[a, b]$, Z_n increases strictly from -1 to 1. Furthermore we have that $Z_n'(y_j) = 0$ for $j = 2, ..., n-1$, and that there exists a unique $c \in (1, a)$ such that $Z_n'(c) = 0$. Rivlin [12] called such a polynomial Z_n a "hard-core" Zolotarev polynomial. Accordingly we say that a polynomial Z_n of exact degree n is a "hard-core" Zolotarev polynomial on $[-1, 1] \cup [a, b]$, $1 < a < b$, if it satisfies conditions (i) and (ii) for some $y_1 := -1, y_2, ..., y_{n-1}, y_n := 1$, where $y_1 < y_2 < \cdots < y_n$. (See Fig. 1.)

An explicit representation of "hard-core" Zolotarev polynomials in terms of elliptic functions has been given by Zolotarev (see [1] or [3]). Let us mention that the solution of the (generalized) problem of Zolotarev with respect to the L_1-norm is much simpler than the solution with respect to the maximum-norm and can be given explicitly without using elliptic functions (see [8] or [10, Corollary 1]).

In 1880 Posse (see [11; 6, pp. 266–268]), like Zolotarev a pupil of Tchebycheff, posed and studied the following problem, now known under his name:

POSSE'S PROBLEM (P). What conditions must the numbers $a, b \in \mathbb{R}$, $1 < a < b$, satisfy in order that there exist an algebraic polynomial $\tilde{P}_n(x) = x^n + \cdots$ minimizing the functional

$$\int_{-1}^{1} |P_n(x)|\, dx + (-1)^n \int_a^b P_n(x)\, dx \tag{1.1}$$

among all monic polynomials P_n of degree n?

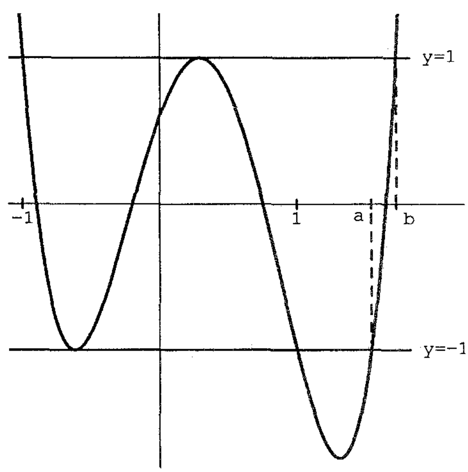

FIG. 1. A "hard-core" Zolotarev polynomial of degree four on $[-1, 1] \cup [a, \ell\ ]$.

In his memoir [11] Posse used elliptic functions and their inverses to express a relation between a and b guaranteeing the existence of a minimum of the above functional. Another existence condition, which is related to the location of zeros of certain orthogonal polynomials, has been given by the author [10, Theorem 4]. Posse also mentioned briefly that, by taking a look at the illustration of a "hard-core" Zolotarev polynomial, one could consider his problem as the analog of Zolotarev's problem in L^1-norm. As it turns out in this paper the connection between these two problems is even closer than one would expect. Indeed as we demonstrate, the solution of Posse's problem is already given by the solution of Zolotarev's problem, more precisely, by the knowledge of "hard-core" Zolotarev polynomials. Furthermore we show that the solution of Posse's problem implies the solutions of a generalized Posse problem, a certain power moment problem, and a Posse related problem for nonnegative polynomials.

2. Main Results

The proof of the main theorem is divided into a series of lemmas, each interesting in its own right. The first lemma can be found in [6, p. 267] in a slightly different form.

Lemma 1. (a) *If there exists a function* $f \in L_1([-1,1] \cup [a,b]) \setminus \mathbb{P}_n$ *such that* $I_f^{(\overset{+}{-})}(p; a, b)$ *attains its minimum on* $\mathbb{P}_n$ *then* $|\int_a^b p| \leqslant \int_{-1}^1 |p|$ *for all* $p \in \mathbb{P}_n$.

(b) *If* $|\int_a^b p| \leqslant \int_{-1}^1 |p|$ *for all* $p \in \mathbb{P}_n$ *then* $I_f^{(\overset{+}{-})}(p; a, b)$ *attains its minimum for every* $f \in L_1([-1,1] \cup [a,b])$.

Lemma 2. *Let* $M \in \mathbb{R}^+$ *and let* $1 \leqslant a < b$. *The following statements are equivalent*:

(a) $\displaystyle\int_a^b |p| \leqslant M \int_{-1}^1 |p|$ *for all* $p \in \mathbb{P}_n$ *and*

$$\int_a^b |\tilde{p}| = M \int_{-1}^1 |\tilde{p}| \qquad \textit{for} \quad \tilde{p} \in \mathbb{P}_n.$$

(b) $\int_a^b p \operatorname{sgn} \tilde{p} - M \displaystyle\int_{-1}^1 p \operatorname{sgn} \tilde{p} = 0$ *for all* $p \in \mathbb{P}_n$

and $\tilde{p}$ *has n simple zeros in* $(-1, 1)$ *and thus no zero in* $[a, b]$. (2.1)

(c) $$\left|\int_a^b p\right| \leqslant M \int_{-1}^1 |p| \qquad \textit{for all} \quad p \in \mathbb{P}_n$$

$$\textit{and} \quad \left|\int_a^b \tilde{p}\right| = M \int_{-1}^1 |\tilde{p}|.$$

Proof. (a) $\Rightarrow$ (b) Set $N(p) = M \int_{-1}^1 |p| - \int_a^b |p|$ for $p \in \mathbb{P}_n$. Since $[N(\tilde{p} + \lambda p) - N(\tilde{p})]/\lambda \geqslant 0$ for all $\lambda \in \mathbb{R}^+$ it follows that

$$\lim_{\lambda \to 0^+} [N(\tilde{p} + \lambda p) - N(\tilde{p})]/\lambda$$

$$= M \int_{-1}^1 p \operatorname{sgn} \tilde{p} - \int_a^b p \operatorname{sgn} \tilde{p} \geqslant 0 \qquad \text{for all} \quad p \in \mathbb{P}_n.$$

Since $-p \in \mathbb{P}_n$ condition (2.1) is proved. Next let us show that $\tilde{p}$ has no zero in (a, b). Let us assume to the contrary that $\tilde{p}$ has a zero in (a, b). Then there exists a $\bar{p} \in \mathbb{P}_n$ such that

$$\operatorname{sgn} \bar{p} = \operatorname{sgn} \tilde{p} \text{ on } (-1, 1) \text{ and } \operatorname{sgn} \bar{p} = \text{const on } (a, b) \text{ and thus}$$

$$M \int_{-1}^1 |\bar{p}| - \int_a^b |\bar{p}| < M \int_{-1}^1 \bar{p} \operatorname{sgn} \tilde{p} - \int_a^b \bar{p} \operatorname{sgn} \tilde{p} = 0,$$

which is a contradiction. Now let us assume that $\tilde{p}$ has $l \leqslant n-1$ sign changes on $(-1, 1)$. Since $\tilde{p}$ has no sign change on (a, b) there exists a $\bar{p} \in \mathbb{P}_n$ such that

$$\operatorname{sgn} \bar{p} = \operatorname{sgn} \tilde{p} \quad \text{on } (-1, 1) \qquad \text{and} \qquad \operatorname{sgn} \bar{p} = -\operatorname{sgn} \tilde{p} \quad \text{on } (a, b).$$

But then (2.1) does not hold for $\bar{p}$, which is the desired contradiction.

(b) $\Rightarrow$ (a) Suppose to the contrary that

$$M^* := \max_{p \in \mathbb{P}_n \setminus \{0\}} \left(\int_a^b |p| \Big/ \int_{-1}^1 |p| \right) = \int_a^b |p^*| \Big/ \int_{-1}^1 |p^*| > M.$$

(The existence of the maximum follows by arguments analogous to those used in the proof of Lemma 3 below.) As in the proof of (a) $\Rightarrow$ (b) we obtain that p^* and M^* satisfy condition (2.1) and that p^* has no zero on (a, b). Hence

$$\int_{-1}^1 p(M^* \operatorname{sgn} p^* \mp M \operatorname{sgn} \tilde{p}) = 0 \qquad \text{for all} \quad p \in \mathbb{P}_n.$$

But this is impossible since $\operatorname{sgn}(M^* \operatorname{sgn} p^* \mp M \operatorname{sgn} \tilde{p}) = \operatorname{sgn} p^*$ and p^* has at most n sign changes.

The implication (b) $\Rightarrow$ (c) follows immediately.

Concerning (c) $\Rightarrow$ (b) we obtain as in the proof of (a) $\Rightarrow$ (b) by Gateaux differentiation that

$$\varepsilon \int_a^b p - M \int_{-1}^1 p \operatorname{sgn} \tilde{p} = 0 \qquad \text{for all} \quad p \in \mathbb{P}_n,$$

where $\varepsilon = \operatorname{sgn}(\int_a^b \tilde{p})$, and that $\tilde{p}$ has n simple zeros in $(-1, 1)$. Hence $\varepsilon = \operatorname{sgn} \tilde{p}$, which proves the assertion. ∎

LEMMA 3. *Let $a \in (1, \infty)$ be given and define for $b \in [a, \infty)$*

$$M(b) := \max_{p \in \mathbb{P}_n \setminus \{0\}} \left(\left| \int_a^b p \right| \Big/ \int_{-1}^1 |p| \right).$$

Then M is a continuous strictly monotone increasing function of b with $M([a, \infty)) = [0, \infty)$.

Proof. Put

$$A = \left\{ (\alpha_0, \alpha_1, ..., \alpha_n) : \int_{-1}^1 \left| \sum_{k=0}^n \alpha_k x^k \right| dx = 1 \right\} \qquad \text{and}$$

$$\|(\alpha_0, \alpha_1, ..., \alpha_n)\|_\infty = \max_{0 \leqslant k \leqslant n} |\alpha_k|.$$

By the equivalence of norms we have that A is bounded and thus, since A contains all its limit points, that A is compact. Hence

$$M(b) = \max_{(\alpha_0, ..., \alpha_n) \in A} \left| \sum_{k=0}^n \alpha_k \int_a^b x^k \, dx \right|$$

is well defined on $[a, \infty)$. Using the boundedness of A we obtain by a rough estimate that for every $(\alpha_0, ..., \alpha_n) \in A$ and $b_1, b_2 \in [a, \infty)$

$$\left| \left| \int_a^{b_1} \sum_{k=0}^n \alpha_k x^k \, dx \right| - \left| \int_a^{b_2} \sum_{k=0}^n \alpha_k x^k \, dx \right| \right| \leqslant K \, |b_2 - b_1|,$$

where $K \in \mathbb{R}^+$, from which it is not difficult to deduce that M is continuous. The monotonicity of M follows immediately with the help of Lemma 2(a). ∎

The next lemma gives a complete characterization of the extremal points of a "hard-core" Zolotarev polynomial.

LEMMA 4. (a) *Let Z_n be a "hard-core" Zolotarev polynomial on $[-1, 1] \cup [a, b]$ and let $-1 =: y_1 < y_2 < \cdots < y_{n-1} < y_n := 1 < y_{n+1} := a < y_{n+2} := b$, where the y_j's, $j = 2, 3, ..., n-1$, denote the zeros of Z_n'.*

Then

$$\sum_{j=1}^{n+2} \varepsilon_j p(y_j) = 0 \qquad \textit{for all} \quad p \in \mathbb{P}_{n-1}, \tag{2.2}$$

where

$$\varepsilon_j = 2(-1)^{n+1-j} \qquad \textit{for} \quad j = 2, 3, ..., n-1,$$
$$\varepsilon_1 = (-1)^n, \qquad \varepsilon_n = -1, \qquad \varepsilon_{n+1} = -1, \qquad \varepsilon_{n+2} = 1.$$

Note that $\sum_{j=1}^{n+2} \varepsilon_j = 0$.

(b) *Suppose that there exist* $(n-2)$ *points* y_j, $j = 2, 3, ..., n-1$, *where* $-1 =: y_1 < y_2 < \cdots < y_{n-1} < y_n := 1 < y_{n+1} := a < y_{n+2} := b$, *such that relation* (2.2) *holds. Then there exists a "hard-core" Zolotarev polynomial* Z_n *on* $[-1, 1] \cup [a, b]$ *with* $Z_n'(y_j) = 0$ *for* $j = 2, 3, ..., n-1$.

Proof. For technical reasons let us first prove part (b).

(b) Let us write condition (2.2) in the form

$$2 \sum_{j=1}^{l^+} (y_j^+)^k - 2 \sum_{j=1}^{l^-} (y_j^-)^k + \sum_{j=1}^{m^+} (x_j^+)^k - \sum_{j=1}^{m^-} (x_j^-)^k = 0$$
$$\text{for} \quad k = 0, ..., n-1; \tag{2.2'}$$

i.e., the points y_j^+, y_j^-, x_j^+, x_j^- denote the y_i's for which ε_i is equal to 2, -2, 1, -1, respectively. Furthermore let us put

$$V^{\binom{+}{-}}(x) = \prod_{j=1}^{l^{\binom{+}{-}}} \left(x - y_j^{\binom{+}{-}}\right) \qquad \text{and} \qquad H^{\binom{+}{-}}(x) = \prod_{j=1}^{m^{\binom{+}{-}}} \left(x - x_j^{\binom{+}{-}}\right).$$

Note that

$$(V^+ V^-)(x) = \prod_{j=2}^{n-1} (x - y_j) \qquad \text{and} \qquad (H^+ H^-)(x) = (x^2 - 1)(x - a)(x - b). \tag{2.3}$$

Using the identity, $k \in \mathbb{N}$,

$$\frac{1}{x-y} = \frac{1}{x} \sum_{\nu=0}^{k} \left(\frac{y}{x}\right)^\nu + \left(\frac{y}{x}\right)^{k+1} \frac{1}{x-y},$$

we obtain in view of (2.2′) that for sufficiently large $|x|$

$$2\frac{(V^+)'(x)}{V^+(x)}-2\frac{(V^-)'(x)}{V^-(x)}+\frac{(H^+)'(x)}{H^+(x)}-\frac{(H^-)'(x)}{H^-(x)}$$
$$=2\left\{\sum_{j=1}^{l^+}\frac{1}{x-y_j^+}-\sum_{j=1}^{l^-}\frac{1}{x-y_j^-}\right\}+\sum_{j=1}^{m^+}\frac{1}{x-x_j^+}-\sum_{j=1}^{m^-}\frac{1}{x-x_j^-}$$
$$=O\left(\frac{1}{x^{n+1}}\right), \tag{2.4}$$

where the first equality easily follows by partial fraction expansion, and thus

$$\ln\left\{\left(\frac{V^+(x)}{V^-(x)}\right)^2\frac{H^+(x)}{H^-(x)}\right\}=O\left(\frac{1}{x^n}\right), \tag{2.5}$$

which implies (note that $2l^+ + m^+ = n = 2l^- + m^-$) that

$$(V^+(x))^2H^+(x)-(V^-(x))^2H^-(x)=\gamma, \tag{2.6}$$

where $\gamma\in\mathbb{R}^-$. Setting

$$\hat{Z}_n(x):=(V^+(x))^2H^+(x)-\gamma/2=(V^-(x))^2H^-(x)+\gamma/2 \tag{2.7}$$

and

$$Z_n(x):=2\hat{Z}_n(x)/(-\gamma) \qquad \text{and} \qquad V_{n-2}(x)=2(V^+V^-)(x)/(-\gamma)$$

we obtain from (2.7) by simple calculation, recalling (2.3),

$$Z_n^2(x)-(x^2-1)(x-a)(x-b)\,V_{n-2}^2(x)=1. \tag{2.8}$$

In view of (2.8) we have that $|Z_n(x)|>1$ on $(1,a)$, $|Z_n(x)|\leqslant 1$ on $[-1,1]\cup[a,b]$ with $|Z_n(y_j)|=1$ for $j\in\{1,...,n+2\}$, and Z_n' vanishes at the $n-2$ zeros y_j, $j\in\{2,...,n-1\}$, of V_{n-2}. From these properties it is not difficult to deduce that Z_n is a "hard-core" Zolotarev polynomial on $[-1,1]\cup[a,b]$ since otherwise Z_n' would have too many zeros.

(a) Put

$$-\gamma/2:=\max_{x\in[-1,1]\cup[a,b]}|\hat{Z}_n(x)|.$$

Then it follows, from the graph of $Z_n(x)$, that $\hat{Z}_n(x)+\gamma/2$ and $\hat{Z}_n(x)-\gamma/2$ have a representation of the form (2.7), where $V^+(x)=\prod_{j=1}^{l^+}(x-y_j^+)$ $(V^-(x)=\prod_{j=1}^{l^-}(x-y_j^-))$ vanishes at the y_j's for which $\hat{Z}_n'(y_j)=0$ and

$\hat{Z}_n(y_j) = -\gamma/2$ $(\gamma/2)$ and $H^+(x) = \prod_{j=1}^{m^+} (x - x_j^+)$ $(H^-(x) = \prod_{j=1}^{m^-} (x - x_j^-))$ vanishes at the y_j's for which $y_j \in \{-1, 1, a, b\}$ and $\hat{Z}_n(y_j) = -\gamma/2$ $(\gamma/2)$. Note that $n = 2l^+ + m^+ = 2l^- + m^-$. Obviously (2.7) implies (2.6), from which (2.5) can be derived. Differentiating (2.5) we obtain (2.4), which is the assertion. ▮

Remark. If we set $\tilde{p}(x) = \prod_{j=2}^{n-1} (x - y_j)$ we obtain by simple calculation that relation (2.2) is equivalent to

$$\int_{-1}^{1} p(x) \cdot \operatorname{sgn} \tilde{p}(x)\, dx - \int_a^b p(x)\, dx = 0 \qquad \text{for all} \quad p \in \mathbb{P}_{n-2}. \tag{2.9}$$

Let us note that Lemma 4 could also be derived from Theorems 1 and 3 of our paper [9]. But in contrast to the above proof, the proofs of Theorems 1 and 3 in [9] are based on previous results of the author on so-called Tchebycheff polynomials on two intervals and are more complicated.

Now we are ready to state our main result, namely that the solution of Posse's problem is already given by the solution of Zolotarev's problem.

THEOREM. *Posse's problem has a solution if and only if there exists a "hard-core" Zolotarev polynomial Z_{n+1} on $[-1, 1] \cup [a^*, b^*]$ with $1 < a^* \leqslant a < b \leqslant b^*$.*

Proof. Necessity. In view of Lemmas 1 and 3 there exists a $b^* \in [b, \infty)$ such that

$$\left|\int_a^{b^*} p\right| \leqslant \int_{-1}^{1} |p| \qquad \text{for all} \quad p \in \mathbb{P}_{n-1} \qquad \text{and} \qquad \left|\int_a^{b^*} \tilde{p}\right| = \int_{-1}^{1} |\tilde{p}|, \tag{2.10}$$

which by Lemma 2 is equivalent to the fact that there exists $\tilde{p} \in \mathbb{P}_{n-1}$ with leading coefficient one which is of the form

$$\tilde{p}(x) = \prod_{j=2}^{n} (x - y_j), \qquad -1 =: y_1 < y_2 < \cdots < y_n < y_{n+1} := 1$$

and satisfies

$$\int_{-1}^{1} p \operatorname{sgn} \tilde{p} - \int_a^{b^*} p = 0 \qquad \text{for all} \quad p \in \mathbb{P}_{n-1}. \tag{2.9}$$

Thus by Lemma 4(b) and the following remark the assertion is proved.

Sufficiency. Since there exists a Zolotarev polynomial Z_{n+1} on $[-1, 1] \cup [a^*, b^*]$ we have by Lemma 4(a), setting $\tilde{p}(x) = Z'_{n+1}(x)/$

$(n+1)(x-c)$, where c is the zero of Z'_{n+1} which lies in (a^*, b^*), and $a=a^*$ and $b=b^*$ that condition (2.9) is fulfilled. Thus with the help of Lemma 2 we obtain that

$$\left|\int_a^b p\right| \leqslant \int_{a^*}^{b^*} |p| \leqslant \int_{-1}^1 |p| \qquad \text{for all} \quad p \in \mathbb{P}_{n-1},$$

which by Lemma 1 proves the sufficiency part. ∎

COROLLARY 1. (a) *If Posse's problem has a solution then*

$$b < \left(2a+1-\cos\frac{\pi}{n+1}\right) \Big/ \left(1+\cos\frac{\pi}{n+1}\right).$$

(b) *Posse's problem is always solvable if*

$$a < b \leqslant \left(3-\cos\frac{\pi}{n+1}\right) \Big/ \left(1+\cos\frac{\pi}{n+1}\right).$$

Proof. If we set

$$\tilde{p}(x) = U_n((2x+1-b)/(b+1))/(x-x_n),$$

where, as usual, U_n is the Tchebycheff polynomial of the second kind of degree n on $[-1, 1]$ and

$$x_n = \frac{b-1}{2} + \frac{(b+1)}{2}\cos\frac{\pi}{n+1}$$

denotes the largest zero of $U_n((2x+1-b)/(b+1))$, then we have

$$\begin{aligned}
&\int_{-1}^{x_n} x^k \operatorname{sgn} \tilde{p}(x)\, dx - \int_{x_n}^{b} x^k \operatorname{sgn} \tilde{p}(x)\, dx \\
&\quad = -\int_{-1}^{b} x^k \operatorname{sgn} U_n((2x+1-b)/(b+1))\, dx = 0 \\
&\quad \text{for} \quad k = 0, ..., n-1. \qquad (2.11)
\end{aligned}$$

(a) Let us assume that Posse's problem has a solution and that $1 < a \leqslant x_n$. Then with the help of (2.11) and by Lemmas 1 and 2 we obtain that

$$\int_{-1}^{a} |\tilde{p}| \leqslant \int_{-1}^{x_n} |\tilde{p}| = \int_{x_n}^{b} |\tilde{p}| \leqslant \int_a^b |\tilde{p}| \leqslant \int_{-1}^1 |\tilde{p}|,$$

which is a contradiction to $a > 1$.

(b) By assumption we have $x_n \leqslant 1$. Hence it follows rom (2.11) and Lemma 2 that for all $p \in \mathbb{P}_{n-1}$

$$\int_a^b |p| \leqslant \int_1^b |p| \leqslant \int_{x_n}^b |p| \leqslant \int_{-1}^{x_n} |p| \leqslant \int_{-1}^1 |p|,$$

which proves by Lemma 1, part (b). ▌

Let us note that Corollary 1(a) implies immediately that there is no interval $[a, b]$, $1 < a < b$, such that Posse's problem has a solution for each $n \in \mathbb{N}$. To state the precise solution of Posse's problem we need elliptic functions.

COROLLARY 2. *Let $n \in \mathbb{N}$ and $a \in (1, \infty)$ be given and choose the modulus k, $0 < k < 1$, such that*

$$dn^2\left(\frac{K}{n+1}, k\right) = \frac{2}{1+a}$$

and put

$$\frac{2}{1+b^*} = cn^2\left(\frac{K}{n+1}, k\right),$$

where as usual $K = \int_0^1 [(1-x^2)(1-k^2x^2)]^{-1/2}\,dx$. Then Posse's problem has a solution if and only if $b \in [a, b^*]$.

Proof. Let $a \in (1, \infty)$ be given. Let us first recall that the following five statements are equivalent:

(i) There exists a unique $b^* \in (a, \infty)$ such that Posse's problem has a solution if and only if $b \in [a, b^*]$.

(ii) Condition (2.10) is fulfilled.

(iii) Condition (2.9) is fulfilled, where $\tilde{p}(x) = \prod_{j=2}^{n} (x - y_j)$ with $-1 < y_2 < \cdots < y_n < 1$.

(iv) Condition (2.2) is fulfilled, where $-1 =: y_1 < y_2 < \cdots < y_n < y_{n+1} := 1$.

(v) There exists a Zolotarev polynomial Z_{n+1} on $[-1, 1] \cup [a, b^*]$.

The first two equivalences follow from Lemmas 1, 2 and 3, the third one by simple calculation, and the fourth one by Lemma 4.

Since by Lemma 3 the existence of such a unique $b^* \in (a, \infty)$ and thus the existence of a Zolotarev polynomial Z_{n+1} on $[-1, 1] \cup [a, b^*]$ is guaranteed we obtain from [1, p. 306] or [3, (19)] that $2/(1+a) =$

$dn^2(K/(n+1), k)$ and $2/(1+b^*)=cn^2(K/(n+1), k)$, where $k^2=(a-1)(b^*+1)/(a+1)(b^*-1)$. Using the fact that by [4, Theorem 1], $sn(K/(n+1), k)$ increases with k, for $0 \leqslant k < 1$, we obtain with the help of the relation $dn^2 u + k^2 sn^2 u = 1$ that $dn(K/(n+1), k)$ is a strictly decreasing function of k, $0 \leqslant k < 1$, and thus k is uniquely determined, which proves the corollary. ∎

Let us note that we have proved in passing that for given $a \in (1, \infty)$ there exists a unique b^* such that there exists a Zolotarev polynomial Z_n on $[-1, 1] \cup [a, b^*]$; this result has been demonstrated by a completely different method, but also without using elliptic functions, in [5, Lemma 1].

Remark. If $b=b^*$, b^* defined as in Corollary 2, then each polynomial $(x-\lambda)\hat{Z}'_{n+1}(x)/(n+1)(x-c)$, $\lambda \in [1, a]$, minimizes the functional (1.1).

Proof. Put $\lambda \in [1, a]$.

$$\tilde{P}_n(x; \lambda) = (x-\lambda)\hat{Z}'_{n+1}(x)/(n+1)(x-c).$$

Then by Lemma 4 and the following remark we have

$$\int_{-1}^{1} x^j \operatorname{sgn} \tilde{P}_n(x; \lambda)\, dx - \int_a^b x^j\, dx = 0 \qquad \text{for} \quad j=0, ..., n-1,$$

which implies that $\tilde{P}_n(\cdot; \lambda)$, $\lambda \in [1, a]$, minimizes the functional (1.1). ∎

A description of the minimizing polynomial for arbitrary $b \in [a, b^*]$ has been given by Posse [11] with the help of elliptic functions—however, he did not recognize the connection with the derivative of the Zolotarev polynomial in the limit case $b=b^*$—and by the author in [10, pp. 255–257] with the help of certain orthogonal polynomials.

Finally, let us show that Posse's problem is equivalent to the following problems:

Problem (A). What conditions must the numbers $a, b \in \mathbb{R}$, $1<a<b$, satisfy in order that the functional

$$I_f^{\overset{+}{(-)}}(p; a, b) := \int_{-1}^{1} |f-p| \underset{(-)}{+} \int_a^b (f-p)$$

attain its minimum on $\mathbb{P}_{n-1}$ for arbitrary $f \in L_1([-1, 1] \cup [a, b])$? As usual $\mathbb{P}_n$ denotes the set of polynomials of degree at most n.

Problem (B). Posse's problem formulated as a power moment problem: What conditions must the numbers $a, b \in \mathbb{R}$, $1<a<b$, satisfy in

order that there exist a piecewise continuous function h with $|h| \leqslant 1$ on $[-1, 1]$ satisfying

$$\int_{-1}^{1} x^k h(x)\, dx = \int_a^b x^k\, dx \qquad \text{for} \quad k = 0, 1, ..., n-1?$$

PROBLEM (C). What conditions must the numbers $a, b \in \mathbb{R}$, $1 < a < b$, satisfy in order that

$$\int_{-1}^{1} T(x) \frac{(a-x)}{\sqrt{(b-x)(a-x)(1-x^2)}}\, dx \geqslant \int_a^b T(x) \frac{(a-x)}{\sqrt{(b-x)(a-x)(1-x^2)}}\, dx$$

for all polynomials T of degree at most $n-1$ which are nonnegative on $[-1, +1]$?

Proof of Equivalence of Problems (A), (B), (C), (P)

(P) $\Leftrightarrow$ (A) This follows immediately from Lemma 1.

(P) $\Leftrightarrow$ (B) *Necessity*. Since $\tilde{P}_n(x) = x^n + \cdots$ is a minimizing polynomial it follows by differentiation that

$$\int_{-1}^{1} x^j \operatorname{sgn} \tilde{P}_n(x)\, dx + (-1)^n \int_a^b x^j\, dx = 0 \qquad \text{for} \quad j = 0, ..., n-1.$$

Setting $h(x) = \pm \operatorname{sgn} \tilde{P}_n(x)$ we prove the implication.

Sufficiency. Obviously (B) implies

$$\left| \int_a^b p \right| = \left| \int_{-1}^{1} ph \right| \leqslant \int_{-1}^{1} |p| \qquad \text{for all} \quad p \in \mathbb{P}_{n-1},$$

which by Lemma 1 gives the assertion.

(B) $\Leftrightarrow$ (C) Let us put

$$s_k := \int_a^b t^k\, dt \qquad \text{for} \quad k \in \mathbb{N}_0.$$

Then for $|x| > b$ we have that

$$\sum_{k=0}^{\infty} \frac{s_k}{x^{k+1}} = \ln\left(\frac{x-a}{x-b}\right)$$

and thus

$$\frac{1}{\sqrt{x^2-1}} \exp\left\{\frac{1}{2}\left(\frac{s_0}{x} + \frac{s_1}{x^2} + \cdots + \frac{s_{n-1}}{x^n}\right)\right\} = \sqrt{\frac{x-a}{x-b}} \frac{1}{\sqrt{x^2-1}} + O\left(\frac{1}{x^{n+1}}\right).$$

Setting

$$F(z)=\frac{z-a}{\sqrt{(z-b)(z-a)(z^2-1)}},$$

where that branch of $\sqrt{\ }$ is chosen for which

$$\lim_{z\to\infty}\frac{z^2}{\sqrt{(z-b)(z-a)(z^2-1)}}=1$$

we have that F is analytic on $\mathbb{C}\backslash([-1,1]\cup[a,b])$ with $\lim_{z\to\infty}F(z)=0$. Hence we obtain from [7, Theorem 4.1 and p. 495] that for $z\in\mathbb{C}\backslash([-1,1]\cup[a,b])$

$$\begin{aligned}F(z)&=\frac{1}{2\pi i}\int_{[-1,1]\cup[a,b]}\frac{F^+(t)-F^-(t)}{t-z}\,dt\\&=\frac{1}{\pi}\int_{-1}^{1}\frac{1}{z-t}\frac{a-t}{\sqrt{(b-t)(a-t)(1-t^2)}}\,dt\\&\quad-\frac{1}{\pi}\int_{a}^{b}\frac{1}{z-t}\frac{a-t}{\sqrt{(b-t)(a-t)(1-t^2)}}\,dt.\end{aligned}$$

Thus finally we obtain

$$\begin{aligned}&\frac{1}{\sqrt{x^2-1}}\exp\left\{\frac{1}{2}\left(\frac{s_0}{x}+\frac{s_1}{x^2}+\cdots+\frac{s_{n-1}}{x^n}\right)\right\}\\&=\frac{\sigma_0}{x}+\frac{\sigma_1}{x^2}+\cdots+\frac{\sigma_{n-1}}{x^n}+O\left(\frac{1}{x^{n+1}}\right)\end{aligned}$$

with

$$\begin{aligned}\pi\sigma_j&=\int_{-1}^{1}t^j\frac{a-t}{\sqrt{(b-t)(a-t)(1-t^2)}}\,dt\\&\quad-\int_{a}^{b}t^j\frac{a-t}{\sqrt{(b-t)(a-t)(1-t^2)}}\,dt.\end{aligned}\tag{2.12}$$

(B) $\Rightarrow$ (C) Since there exists a piecewise continuous function h with $|h|\leqslant 1$ on $[-1,1]$ such that

$$\int_{-1}^{1}t^jh(t)\,dt=\int_a^b t^j\,dt\qquad\text{for}\quad j=0,...,n-1,$$

it follows from [2, Theorem 4 and p. 69] that the sequence $\sigma_0,\sigma_1,...,\sigma_{n-1}$

defined by (2.12) is nonnegative definite on $[-1, 1]$. Hence, by definition, $\sum_{j=0}^{n-1} A_j \sigma_j \geqslant 0$ for every polynomial $T(x) = \sum_{j=0}^{n-1} A_j x^j$ of degree $n-1$ which is nonnegative on $[-1, 1]$, which proves the necessity part.

(C) $\Rightarrow$ (B) Since condition (C) is equivalent to the nonnegative definiteness of $\sigma_0, \sigma_1, ..., \sigma_{n-1}$ it again follows by [2, Theorem 4 and p. 69] that there exists a piecewise continuous function h such that

$$\int_{-1}^{1} t^j h(t)\, dt = s_j \qquad \text{for} \quad j = 0, ..., n-1,$$

which proves the implication.

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